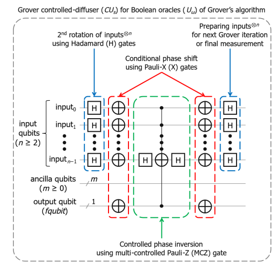
Nov 30, 2024 Version 2

Grover controlled-diffuser (CU_s) for quantum Boolean oracles of Grover's algorithm V.2



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Protocol status: Working
We use this protocol and it's working

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Disclaimer

Abstract

The Grover controlled-diffuser (CU_S) for quantum Boolean oracles (U_ω) is introduced as a new approach for Grover's algorithm [1-3], to search for all solutions for arbitrary logical structures of such oracles, since the standard Grover diffuser (U_S) [1-3] is not able to find all correct solutions for some logical structures of U_ω.

This protocol constructs the quantum circuit of the CU_S operator [4] of Grover's algorithm, which relies on the states of the output qubit (as the reflection of Boolean decisions from a U_ω) without relying on the conventional phase kickback mechanism. The CU_S operator successfully searches for all correct solutions for all U_ω regardless of their different logical structures, such as POS, SOP, ESOP, CSP-SAT, XOR-SAT, just to name a few.

Troubleshooting

Preliminary Notes

1

Note

Grover’s algorithm [1-3] is the most well-known quantum search algorithm that:

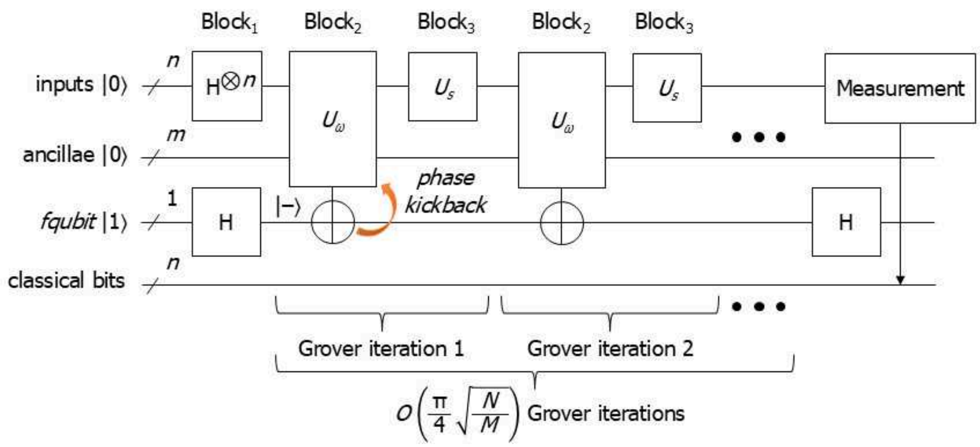
1. Finds solutions for both Boolean and Phase oracles in quadratic speedup.
2. Constructs other quantum algorithms, such as the quantum counting algorithm [5, 6].

2

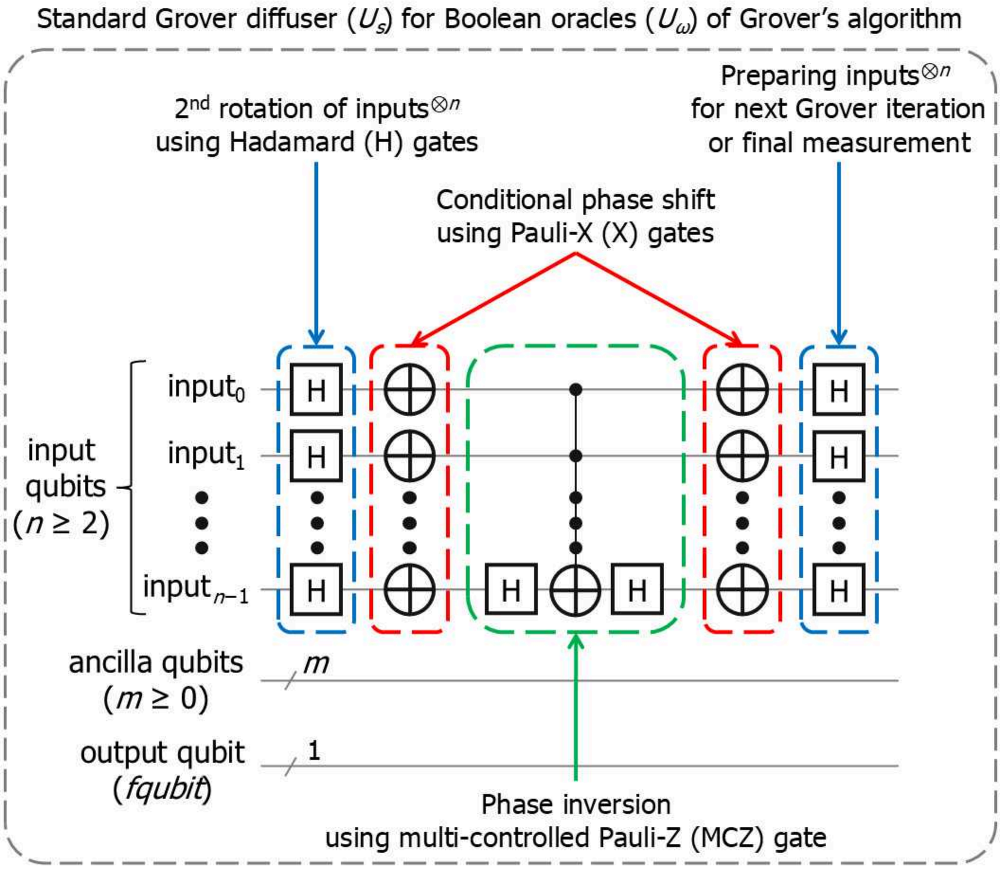
Note

In general, Grover's algorithm consists of three components (Blocks), as illustrated below:

1. $Block_1$ initializes n input qubits to a uniform distribution using Hadamard (H) gates, i.e., generates a complete quantum search space of $\{|0\rangle, |1\rangle\}^{\otimes n}$ for Grover's algorithm to search for solutions (marked elements).
2. $Block_2$ consists of a Boolean or Phase oracle (U_ω) [1-4] that inverts the phase of marked elements, as the "first rotation of solutions" over the complete quantum search space. Such phase inversion occurs due to the phase kickback for a Boolean oracle or the effect of quantum phase-based gates on n input qubits for a Phase oracle.
3. $Block_3$ consists of the Grover diffusion operator (U_s) that performs the "second rotation of solutions", conditional phase shift, and phase inversion, by rotating and amplifying the amplitudes of the marked elements from $Block_2$, as the final found solutions, as demonstrated below.



Schematic of Grover's algorithm to solve a Boolean oracle (U_ω) using the standard Grover diffusion operator (U_s), for a number of Grover iterations (loops). Please observe that both $Block_2$ and $Block_3$ are treated as one Grover iteration.



The quantum circuit of the standard Grover diffusion operator (U_s).

3

Note

1. In the quantum domain, an oracle (U_ω) is the conceptual expression of a problem in the classical domain.
2. A U_ω can be constructed as a Boolean oracle (using quantum Boolean-based gates) or a Phase oracle (using quantum phase-based gates).
3. Grover's algorithm solves a U_ω ($Block_2$) using the U_s operator ($Block_3$), which searches for one solution in the evaluation complexity of $O(\sqrt{N})$ or for a number of solutions in the evaluation complexity of $O\left(\frac{\pi}{4}\sqrt{\frac{N}{M}}\right)$ for $M < N/2$ as an algorithmic constraint, where:

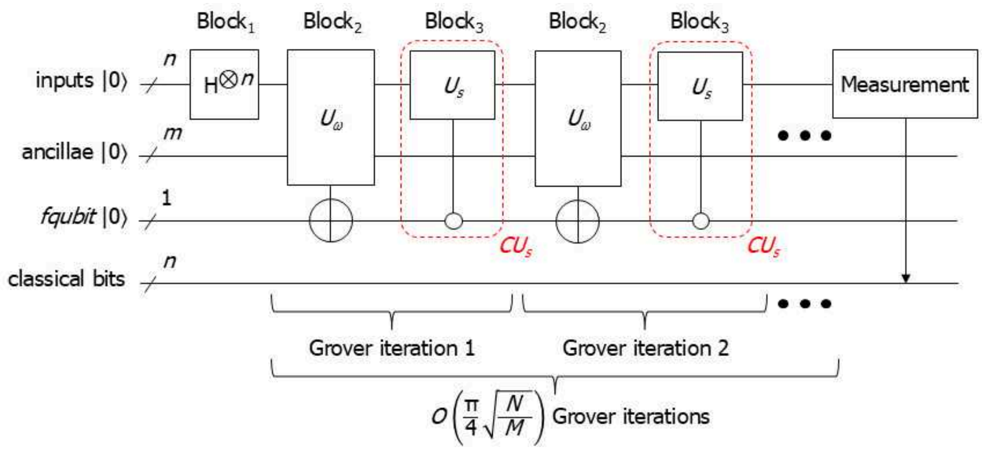
- $N = 2^n$.
- n is the total number of input qubits for a U_ω.
- M is the total number of solutions for a U_ω, i.e., the solutions of an expressed problem as a U_ω.

Please observe that the U_s operator is designed to rotate and amplify the amplitudes of N by their average (*inversion about average* [1-3]), and when more than half of quantum search space ($\{|0\rangle, |1\rangle\}^{\otimes n}$) is filled by M, Grover's algorithm makes random guesses of marked and unmarked elements as solutions!

4

Note

Our Grover controlled-diffuser (CU_s) is introduced as a new approach to overcome the algorithmic constraint of $M < N/2$, by controlling the operation of U_s using the output qubit (*fqubit*) of a Boolean oracle (U_ω), without relying on the conventional phase kickback mechanism, as shown below. Such that, the CU_s operator is designed for Boolean oracles only, since Phase oracles do not have any output qubit.

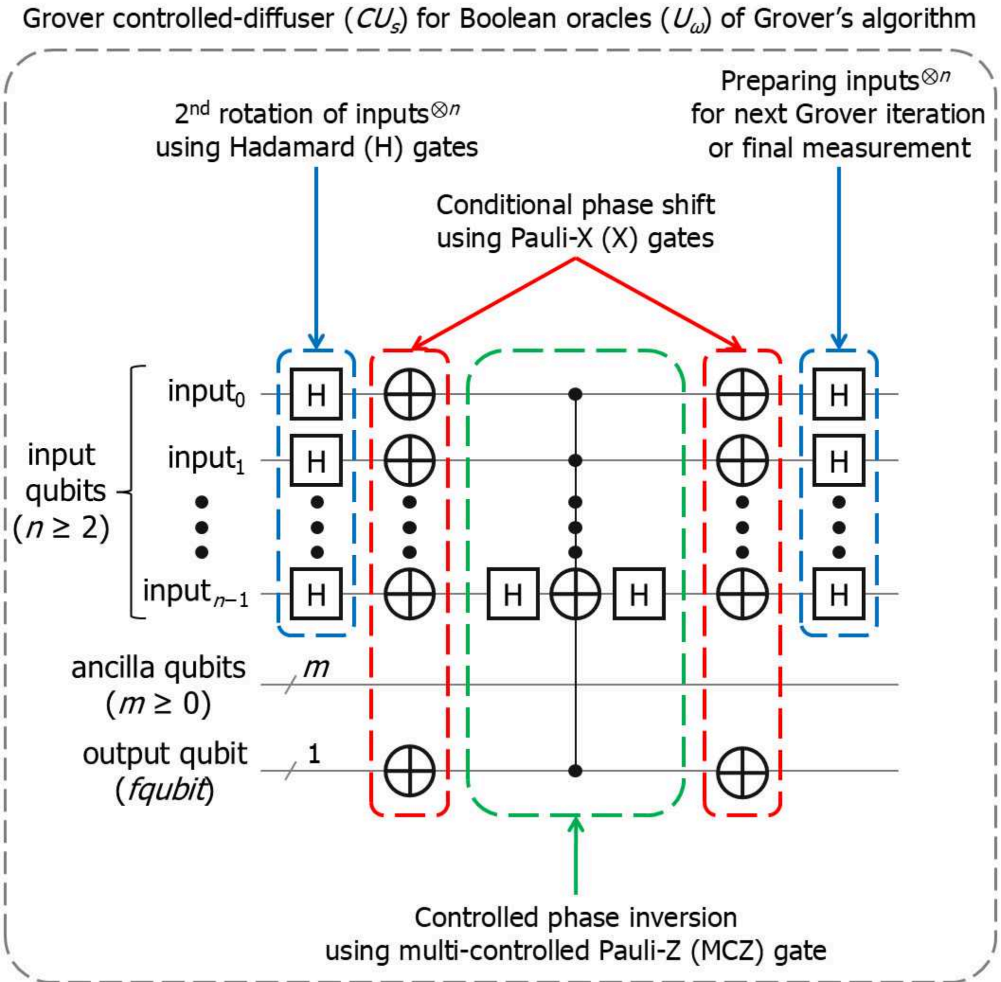


Schematic of Grover's algorithm to solve a Boolean oracle (U_ω) using our Grover controlled-diffusion operator (CU_s), for a number of Grover iterations (loops). Note that both $Block_2$ and $Block_3$ are treated as one Grover iteration.

Please observe that *fqubit* is the "functional qubit" as the one ancilla output qubit for a Boolean oracle (U_ω). When the state of *fqubit* = $|1\rangle$, a solution is found by a U_ω, and there is no need to activate the quantum operation of CU_s, i.e., a solution is passed through and $CU_s \equiv I$. Otherwise, when the state of *fqubit* = $|0\rangle$, a non-solution is found by a U_ω, and the quantum operation of CU_s is activated to search for any remaining solutions (residues) [4], i.e., $CU_s \equiv U_s$.

Hence, the states of *fqubit* instruct the quantum operations of CU_s to search for all correct solutions, as stated in the following algebraic formula and demonstrated in the figure below.

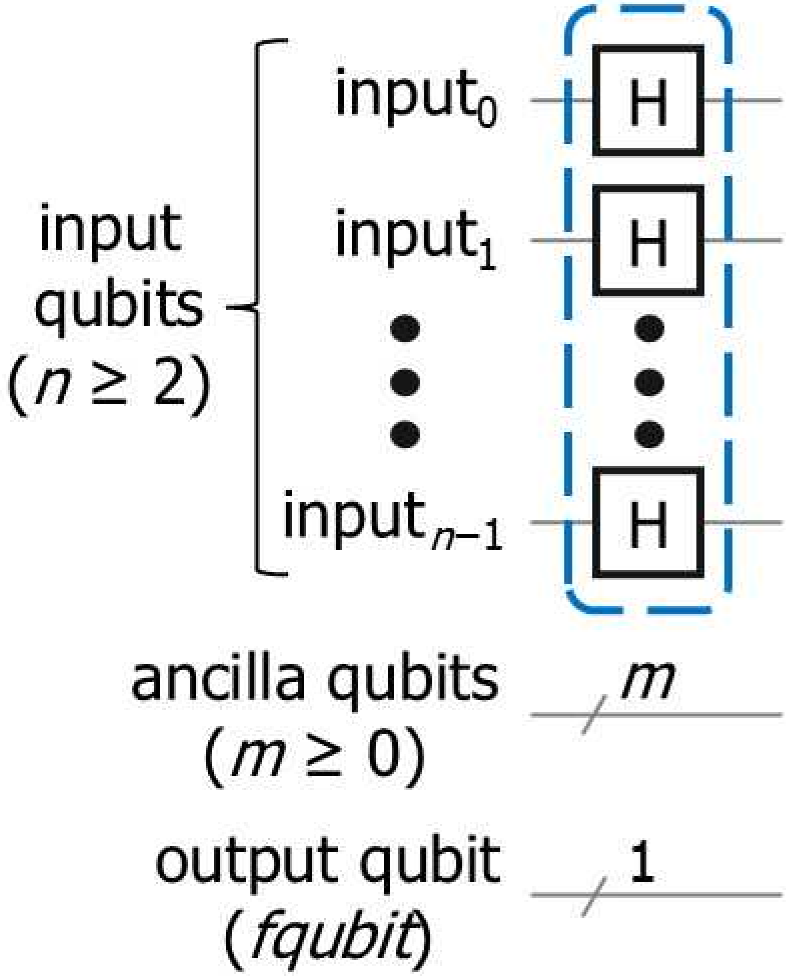
$$CU_s = U_{s\ (\ fqubit\ =\ |0\rangle\ )} + I_{(\ fqubit\ =\ |1\rangle\ )}$$



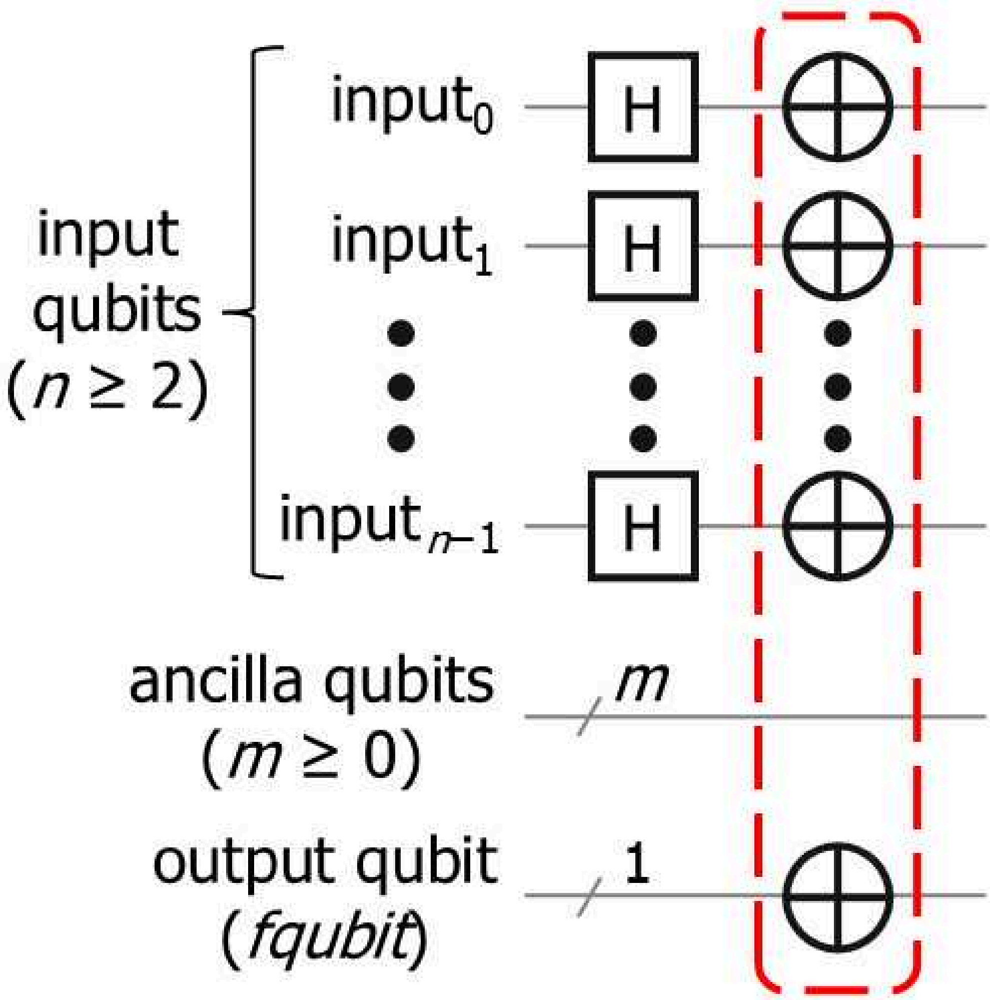
The quantum circuit of our Grover controlled-diffusion operator (CU_s).

The CU_s Protocol (for Boolean oracles of Grover's algorithm)

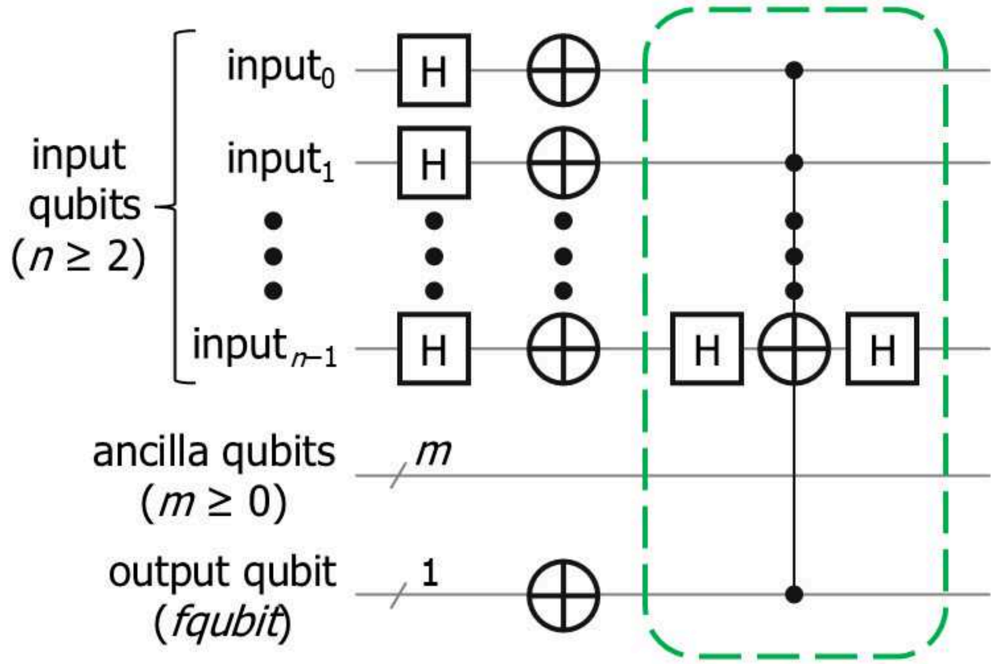
5 Rotate all n input qubits of a Boolean oracle (U_ω), as the "second rotation of solutions", using n Hadamard (H) gates, where $n \geq 2$. Note that all m ancilla qubits including the *fqubit* do not require such a rotation, where $m \geq 0$.



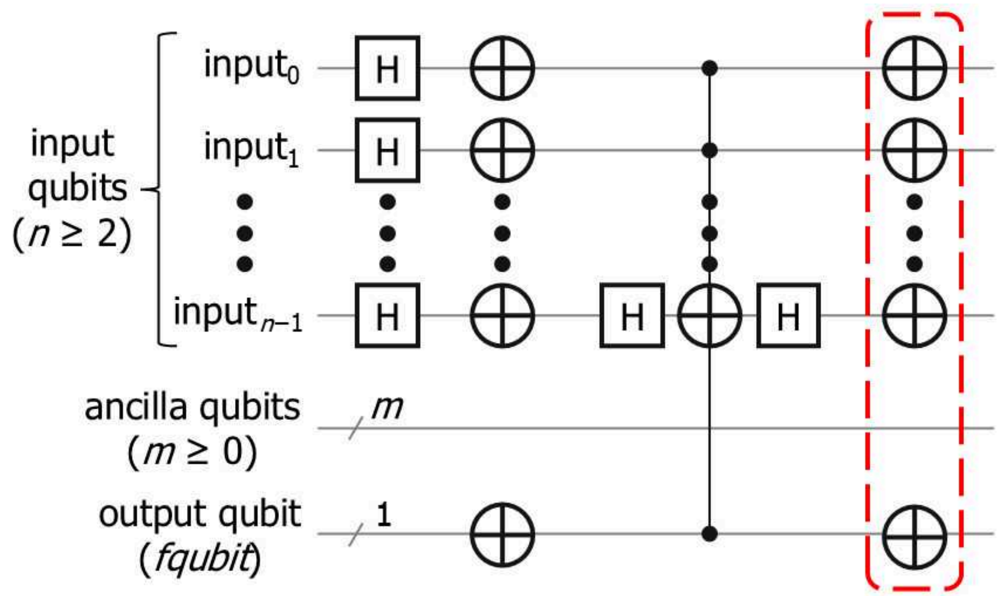
6 Conditionally shift the phases of all *n* input qubits and *fqubit*, using *n*+1 Pauli-X (X) gates. Note that all *m* ancilla qubits are not included for such a conditional phase shift.



7 Invert the phases of all *n* input qubits depending on the inverted states of *fqubit*, using one multi-controlled Pauli-Z (MCZ) gate of *n*+1 qubits. Note that all *m* ancilla qubits are not included for such a controlled phase inversion.

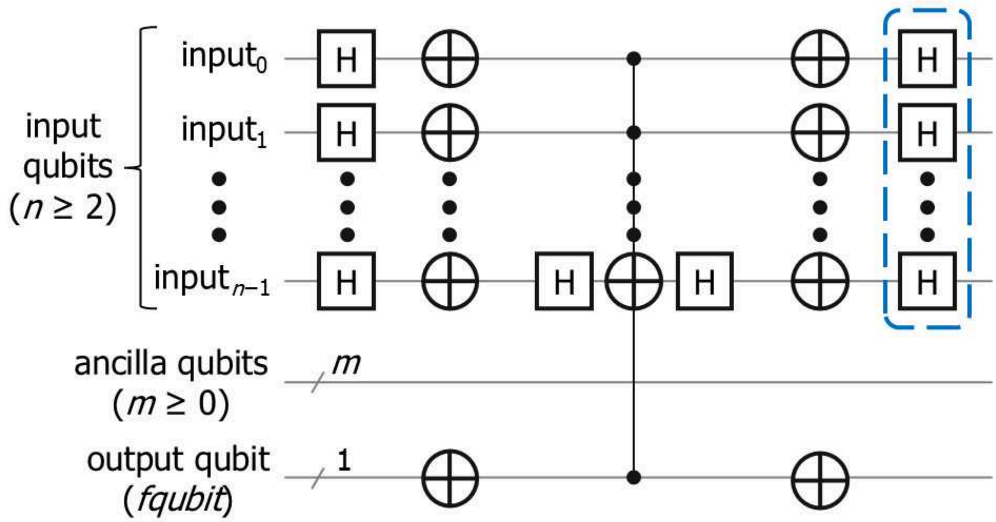


8 Uncompute (mirror) the aforementioned step of the conditional phase shift, using n+1 X gates.



9 Finally, uncompute (mirror) the aforementioned step of the "second rotation of solutions", using n H gates, to prepare all n inputs qubits for the next Grover iteration (if

required) or the final measurement.



The Quantum Cost of CU_s Operator

10 For a Boolean oracle (U_ω) of Grover's algorithm consisting of n input qubits, m ancilla qubits, and one *fqubit,* the quantum cost of CU_s operator that defines the total utilized number of standard quantum gates is stated as follows, where $n > 2$ and $m > 0$.

$$Quantum\ Cost_{\ CU_S} = (2n+2)\ H\ +\ (2n+2)\ X\ +\ MCX_{n+1}$$

Note that MCX_{n+1} is a multi-controlled Pauli-X gate of n+1 qubits, which is the (n+1)-bit Toffoli gate of n controls and one target.

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