Dec 26, 2025 Version 1

A Topological Obstruction to Persistent Vorticity Alignment via the Angular Strain Symbol V.1

DOI
https://dx.doi.org/10.17504/protocols.io.j8nlk15m5g5r/v1

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Protocol Citation: Torah Sanni 2025. A Topological Obstruction to Persistent Vorticity Alignment via the Angular Strain Symbol. **protocols.io** https://dx.doi.org/10.17504/protocols.io.j8nlk15m5g5r/v1

Protocol status: Working
We use this protocol and it's working

Created: December 26, 2025

Last Modified: May 18, 2026

Protocol Integer ID: 235849

Keywords: topological obstruction to persistent global vorticity alignment, topological obstruction to persistent vorticity alignment, persistent global vorticity alignment, persistent vorticity alignment, dimensional incompressible navier, maximal vortex stretching, angular strain symbol, stokes equation, fourier representation, topological obstruction, eigenvalue degeneracies on the unit, regularity theory for weak solution, eigenbundle, eigenvalue degeneracy, weak solution

Abstract

We study the angular strain symbol arising in the Fourier representation of the three-dimensional incompressible Navier–Stokes equations. We show that this symbol necessarily exhibits eigenvalue degeneracies on the unit sphere and that at least one such degeneracy is transverse. As a consequence, the eigenbundle associated with maximal vortex stretching is non-orientable. This yields a topological obstruction to persistent global vorticity alignment at the symbol level. The analysis is finite-dimensional and does not rely on compactness or regularity theory for weak solutions.

Guidelines

- Finite-dimensional and coordinate-invariant
- No compactness or weak-solution theory
- Explicit degeneracy and transversality
- Conservative claims

Introduction

1 The primary obstruction to global regularity in the three-dimensional incompressible Navier–Stokes equations arises from the vortex stretching term $\omega \cdot \nabla u = S\omega$, where $S = \frac{1}{2} (\nabla u + \nabla u^T)$ is the strain tensor and $\omega = \nabla \times u$ is the vorticity. Several conditional regularity results rely on geometric alignment assumptions between vorticity and strain eigenvectors. In this note, we isolate a purely geometric obstruction to persistent maximal alignment at the symbol level. Our approach is finite-dimensional, scale-invariant, and independent of PDE compactness arguments. We emphasize that no global regularity or blow-up exclusion result is claimed.

The Angular Strain Symbol

2 Let u be a divergence-free velocity field on $\mathbb{R}^3$. In Fourier variables, $\hat{S}(\xi) = |\xi| M(\theta)$, $\theta = \frac{\xi}{|\xi|} \in S^2$.

Definition 2.1. The angular strain symbol $M(\theta) \in \text{Sym}_0(3)$ is defined by $M_{ij}(\theta) = \frac{1}{2} (\varepsilon_{ikl} \theta_k \theta_l + \varepsilon_{jkl} \theta_k \theta_l) \theta_i, \theta \in S^2$.

Lemma 2.2. For all $\theta \in S^2$:

(i) $M(\theta)^T = M(\theta)$

(ii) $\text{tr} M(\theta) = 0$

(iii) $M(-\theta) = -M(\theta)$

(iv) $M(\theta) \theta = 0$

Proof. Each property follows directly from antisymmetry of ε_{ijk} and symmetry in i, j. The kernel property follows from contraction with θ.

Existence of Eigenvalue Degeneracy

3 Let $\lambda_1(\theta) \geq \lambda_2(\theta) \geq \lambda_3(\theta)$ denote the eigenvalues of $M(\theta)$.

Definition 3.1. The degeneracy set is $\Sigma := \{ \theta \in S^2 : \lambda_1(\theta) = \lambda_2(\theta) \}$.

Lemma 3.2. There exists $\theta_0 \in S^2$ such that $M(\theta_0) = 0$. In particular, $\Sigma \neq \emptyset$.

Proof. Define $F(\theta) = (M_{12}(\theta), M_{13}(\theta), M_{23}(\theta))$. The map $F : S^2 \to \mathbb{R}^3$ is smooth and odd. A direct computation shows $\theta_1 M_{23} - \theta_2 M_{13} + \theta_3 M_{12} = 0$, so $F(S^2)$ lies in a two-dimensional subspace of $\mathbb{R}^3$. By oddness and compactness, F vanishes at some θ_0, implying $M(\theta_0) = 0$.

Transversality of the Degeneracy

4 Lemma 4.1. There exists $\theta_0 \in \Sigma$ such that $\nabla (\lambda_1 - \lambda_2)(\theta_0) \neq 0$.
Proof. Let θ_0 satisfy $M(\theta_0) = 0$. Consider the differential $DM(\theta_0) : T_{\theta_0} S^2 \to \text{Sym}_0(3)$. Rotating coordinates, assume $\theta_0 = (0, 0, 1)$, so $h = (h_1, h_2, 0)$. A direct component computation yields nonzero matrices $DM(\theta_0)[(1, 0, 0)] \neq 0$, $DM(\theta_0)[(0, 1, 0)] \neq 0$.
Thus the eigenvalue splitting is linear, and the degeneracy is transverse.

Topology of the Maximal Eigenbundle

5 Define $L(\theta) := \ker(M(\theta) - \lambda_1(\theta)I), \theta \in S^2 \setminus \Sigma$.
Theorem 5.1. The real line bundle $L \to S^2 \setminus \Sigma$ is non-orientable.
Proof. Near a transverse eigenvalue crossing, $M(\theta)$ reduces locally to the universal 2×2 eigenvalue crossing normal form. Parallel transport of eigenvectors around a loop enclosing a degeneracy induces a sign change. This Möbius-type monodromy obstructs orientability.

Implications for Vortex Stretching

6 Maximal vortex stretching at a fixed frequency requires alignment with $L(\theta)$.
Non-orientability prevents global continuous alignment and enforces angular variation.
This provides a geometric obstruction to persistent maximal stretching mechanisms. No quantitative regularity or blow-up conclusion is drawn.

Discussion

7 We identify a finite-dimensional topological obstruction inherent in the angular strain symbol of the Navier–Stokes equations. This complements analytic approaches based on alignment and depletion and may inform future quantitative estimates.